ON MANIFOLDS OF SMALL DEGREE

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ABSTRACT. Let $X \subset \mathbb{P}^n$ be a complex projective manifold of degree d and arbitrary dimension. The main result of this paper gives a classification of such manifolds (assumed moreover to be connected, non-degenerate and linearly normal) in case $d \leqslant n$. As a by-product of the classification it follows that these manifolds are either rational or Fano. In particular, they are simply connected (hence regular) and of negative Kodaira dimension. Moreover, easy examples show that $d \leqslant n$ is the best possible bound for such properties to hold true. The proof of our theorem makes essential use of the adjunction mapping and, in particular, the main result of [14] plays a crucial role in the argument.

1. INTRODUCTION

Let $X \subset \mathbb{P}^n$ be a complex connected projective manifold of dimension r and degree d. Assume moreover that X is non-degenerate and $d \leqslant n$. The results contained in this paper have the following topological consequence:

$(*)$ *If X is as above, then X is simply connected*

The bound $d \leqslant n$ is optimal for the validity of $(*)$. Indeed, there exist r-dimensional elliptic scrolls in $\mathbb{P}^{2r}$, of degree $2r+1$ (see [13], 5.2); they have $b_1 = 2$.

To the best of our knowledge, $(*)$ was not even conjectured before. There was, however, the following question raised by F.L. Zak:

Is a linearly normal r-dimensional manifold in $\mathbb{P}^{2r+1}$ of degree $\leqslant 2r+1$ and whose embedded secant variety equals $\mathbb{P}^{2r+1}$, a regular variety (i.e. having $b_1 = 0$)?

We refer the interested reader to [1] for a pertinent discussion about the relevance of Zak's question. It follows from $(*)$ that the answer to this question is positive, even under more general assumptions.

We would like to mention also two related topological ancestors of $(*)$. The first one is (a special case of) Barth-Larsen's theorem (see [3] and, for a singular

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version, [8]):

(B-L) *If* $r \geqslant \operatorname{codim}_{\mathbb{P}^n}(X)+1$, *then* $\pi_1(X)=(0)$.

The second result is Gaffney-Fulton-Lazarsfeld's theorem about branched coverings of $\mathbb{P}^r$ (see [9, 8]):

(G-F-L) *If* $X \to \mathbb{P}^r$ *is a normal finite covering of degree* $d \leqslant r$, *then* $\pi_1(X)=(0)$.

Note that, for $d \leqslant r$, $(*)$ follows either from (B-L) or from (G-F-L). We refer to [8] for a very nice discussion of such topological aspects. Recall that the Δ-genus of X in $\mathbb{P}^n$ is, by definition, the non-negative integer $\Delta := d + r - h^0(X, \mathcal{O}_X(1))$. Assuming X to be linearly normal in $\mathbb{P}^n$ (which is not restrictive), condition $d \leqslant n$ may be restated as:

$$r \geqslant \Delta + 1.$$

So we see that $(*)$ is a Barth-Larsen-type result in which codimension is replaced by Δ-genus.

Our proof of $(*)$ is, however, not topological. We deduce $(*)$ from the following geometric result:

$(**)$ *If* X *is as above, then either:*
(1) $b_2 = 1$ *and* X *is a Fano manifold, or*
(2) $b_2 \geqslant 2$ *and* X *is rational.*

It is well-known that both rational and Fano manifolds are simply-connected; see [16] for a far-reaching common generalization. So $(*)$ follows from $(**)$. The first case in $(**)$ may be seen as generic, as it includes all complete intersections of dimension at least three. Indeed, we shall prove:

$(***)$ *Manifolds with* $d \leqslant n$ *and* $b_2 \geqslant 2$ *may be classified completely. There are* 6 *infinite series (having arbitrarily large dimension and degree) and* 14 *"sporadic" examples. Moreover, all turn out to be rational.*

The precise list is given in the statement of the main result, see the next section.

The proof of the main theorem will occupy Section 4. It relies on a very detailed study of the adjunction mapping (see e.g. [4], Chapters 9–11 for a complete treatment). Moreover, the main result of [14] plays a key role in the proof. We note that, besides classical adjunction theory, some nontrivial facts coming from Mori theory are also used in [14]. Finally, the classification of manifolds of small Δ-genus (cf. [6], [7], [12]) is also needed.

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2. STATEMENT OF THE MAIN RESULT

Our main result is the following:

Theorem. *Let $X \subset \mathbb{P}^n$ be a connected projective manifold over $\mathbb{C}$, of dimension r and degree d. Assume moreover that X is non-degenerate and linearly normal. If $d \leqslant n$, then one of the following holds:*

(i) $r \geqslant 1$, *X is Fano,* $b_2(X) = 1$*;*

(ii) *X is Fano and either:*

(a) $2 \leqslant r \leqslant 4$, $3 \leqslant d \leqslant 8$, *$X$ is a classical del Pezzo manifold with* $b_2(X) \geqslant 2$ (cf. Theorem B below)*;*

(b) $r = 3$, $d = 9$, *X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{F}_1$, where $\mathbb{F}_1$ is the blowing-up of $\mathbb{P}^2$ in a point, embedded in $\mathbb{P}^4$ as a rational scroll of degree* 3*;*

(c) *X is one of the following scrolls over $\mathbb{P}^2$:* $r = 4$, $d = 10$, $X \simeq \mathbb{P}(T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ *or* $r = 4$, $d = 11$, $X \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ *or* $r = 5$, $d = 10$, *X is the Segre embedding of* $\mathbb{P}^2 \times \mathbb{P}^3$*;*

(iii) $r \geqslant 2$, $d \geqslant r$, *X is a scroll over $\mathbb{P}^1$ (i.e. a linear section of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^m$);*

(iv) $r \geqslant 3$ *and there is a vector bundle $\mathcal{E}$ over $\mathbb{P}^1$, of rank $r+1$ and of splitting type $e = (e_0, \ldots, e_r)$, such that, if L denotes the tautological divisor on $\mathbb{P}(\mathcal{E})$ and F denotes a fibre of the projection $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$, X embeds in $\mathbb{P}(\mathcal{E})$, $L|_X = H$ and either:*

(a) $n = d = 2r - 1$, $e = (1, \ldots, 1, 0, 0)$, $X \in |2L + F|$*;*

(b) $n = d = 2r$, $e = (1, \ldots, 1, 0)$, $X \in |2L|$*;*

(c) $n = d = 2r + 1$, $e = (1, \ldots, 1)$, $X \in |2L - F|$*;*

(d) $r \geqslant 4$, $n = 2r+1$, $d = 2r$, $e = (1, \ldots, 1)$, $X \in |2L - 2F|$ *or, equivalently,* $X \simeq \mathbb{P}^1 \times Q^{r-1}$ *embedded Segre;*

(e) $n = d = 2r + 2$, $e = (1, \ldots, 1, 2)$, $X \in |2L - 2F|$.

Remarks. (i) Except for case (i), all manifolds appearing in the theorem are rational.

(ii) All cases listed really occur.

(iii) An inspection of the above list (or a direct argument) shows that if we assume $d \leqslant r$, X is either the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{r-1}$ in $\mathbb{P}^{2r-1}$ or a Fano manifold with $b_2 = 1$. In case $d \leqslant r - 2$ all examples I know of are complete intersections.

(iv) Manifolds from case (iv) (b) up to (iv) (e) in the theorem are also Fano.

3. Conventions and prerequisites

We follow the customary notation in Algebraic Geometry (see e.g. [11]). We denote by $X \subset \mathbb{P}^n_{\mathbb{C}}$ a complex projective connected manifold. We let d be its degree and r its dimension; $s = n - r$ is the codimension of X in $\mathbb{P}^n$. The irregularity of X is by definition $q =: h^1(X, \mathcal{O}_X)$. H will denote a hyperplane section of $X \subset \mathbb{P}^n$. The sectional genus of X, denoted g, is the genus of the curve $X \cap H_1 \cap \cdots \cap H_{r-1}$, where $H_1, \ldots, H_{r-1}$ are generic hyperplanes in $\mathbb{P}^n$. Adjunction formula reads:

$$2g - 2 = (K + (r-1)H)H^{r-1},$$

where K is a canonical divisor for X.

The Δ-genus of X is by definition

$$\Delta = d + r - h^0(X, \mathcal{O}_X(H))$$

and is a non-negative integer.

X is said to be a *scroll* over the manifold Y if $X \simeq \mathbb{P}(\mathcal{E})$ for some vector bundle $\mathcal{E}$ on Y, such that $\mathcal{O}_X(H)$ identifies to the tautological line bundle of $\mathbb{P}(\mathcal{E})$.

X is said to be a *hyperquadric fibration* over the smooth curve C if there is a morphism $\pi : X \to C$ such that the fibres of π are hyperquadrics with respect to the embedding induced by $\mathcal{O}_X(H)$. It turns out that singular fibres of π are ordinary cones (see [12]). In the sequel, we denote by Q^r a hyperquadric of dimension r.

The adjunction mapping of X, denoted below by φ, is the rational map on X associated to the linear system $|K + (r-1)H|$. See e.g. [4], Chapters 9–11 for a complete study of its properties.

We recall two results on the classification of manifolds of small Δ-genus. The first one is classical (see e.g. [12], Proposition 2.3).

Theorem A. *The following are equivalent:*

(i) $\Delta = 0$*;*

(ii) $g = 0$*;*

(iii) X *is either* $\mathbb{P}^r$*,* $H \in |\mathcal{O}_{\mathbb{P}^r}(1)|$ *or the hyperquadric* $Q^r \subset \mathbb{P}^{r+1}$ *or* $\mathbb{P}^2$*,* $H \in |\mathcal{O}_{\mathbb{P}^2}(2)|$ *or a scroll over* $\mathbb{P}^1$*.*

The next result is due to del Pezzo if $r = 2$, to Fano and Iskovskih for $r = 3$ and to Fujita in general (see also [12], Proposition 2.4 for some other characterisations).

Theorem B. (Fujita, [6], [7]) *Assume that* $r \geqslant 2$*. The following are equivalent:*

(i) $\Delta = 1$*;*

(ii) X *is either a classical del Pezzo surface (anticanonical embedding of either* $\mathbb{P}^1 \times \mathbb{P}^1$ *or of the blowing-up of* $\mathbb{P}^2$ *in at most six points) or , if* $r \geqslant 3$*, one of the following: a hypercubic, a complete intersection of type* $(2,2)$*, a linear section of*

the Plücker embedding of the Grassmannian of lines in $\mathbb{P}^4$*, the Segre embedding of* $\mathbb{P}^2 \times \mathbb{P}^2$*, its hyperplane section (which is* $\mathbb{P}(T_{\mathbb{P}^2})$*), the Segre embedding of* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$*, the scroll over* $\mathbb{P}^2$*,* $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ *(this is the blowing-up of* $\mathbb{P}^3$ *at a point), or the Veronese embedding* $v_2(\mathbb{P}^3)$*.*

Recall that X is a Fano manifold if $-K$ is ample. We see that the examples listed in Theorem B (which were called *classical del Pezzo manifolds* in [12]) are special Fano manifolds.

4. Proof of the theorem

We begin with the following simple fact.

Lemma 1. *Let* C *be a smooth projective curve of positive genus and let* $\mathcal{L} \in \mathrm{Pic}(C)$ *with* $\deg(\mathcal{L}) > 0$*. Then we have* $h^0(\mathcal{L}) \leqslant \deg(\mathcal{L})$*.*

Proof. If $\mathcal{L}$ is special, we may apply Clifford's theorem. If $\mathcal{L}$ is non-special, the result follows from the Riemann-Roch theorem. □

Proposition 2. *Let* C *be a smooth projective curve of positive genus and let* $\mathcal{E}$ *be an ample and spanned vector bundle on* C*. Then we have* $h^0(\mathcal{E}) \leqslant \deg(\mathcal{E})$*.*

Proof. We proceed by induction on $e =: \mathrm{rank}(\mathcal{E})$. When $e = 1$, we may apply Lemma 1. Assume now $e \geqslant 2$. As $\mathcal{E}$ is ample and spanned, it follows that $h^0(\mathcal{E}) > e$. So, for $p \in C$, we may find a non-zero section $s \in H^0(C, \mathcal{E}(-p))$. s induces an exact sequence:

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow 0,$$

where $\mathcal{L} \in \mathrm{Pic}(C)$, $\deg(\mathcal{L}) =: l > 0$, and $\mathcal{E}'$ is ample, spanned and of rank $e - 1$. We have

$$\deg(\mathcal{E}) - l = \deg(\mathcal{E}') \geqslant h^0(\mathcal{E}') \geqslant h^0(\mathcal{E}) - h^0(\mathcal{L})$$

by the induction hypothesis and the cohomology sequence of the above exact sequence. Applying once again Lemma 1 we get $\deg(\mathcal{E}) \geqslant h^0(\mathcal{E})$. □

Corollary 3. *Let* $X \subset \mathbb{P}^n$ *be a scroll over a smooth curve* C*. Assume that* X *is non-degenerate and* $d \leqslant n$*. Then* $C \simeq \mathbb{P}^1$*.*

Proof. Let $X \simeq \mathbb{P}(\mathcal{E})$. If $g(C) > 0$, by Proposition 2 we get

$$n + 1 \leqslant h^0(X, \mathcal{O}_X(H)) = h^0(C, \mathcal{E}) \leqslant \deg(\mathcal{E}) = d,$$

a contradiction. □

Lemma 4. *Let $X \subset \mathbb{P}^{n=r+s}$ be smooth connected non-degenerate with $d \leqslant n$. Assume moreover that $r \leqslant s+1$. Then we have:*
(i) $g \leqslant r-1$*; and*
(ii) $d \geqslant 2g+1$.

Proof. (i) Let $C \subset \mathbb{P}^{s+1}$ be the curve-section of X. If H_C is special, by Clifford's theorem we get

$$s+2 \leqslant h^0(C, \mathcal{O}_C(H_C)) \leqslant \frac{d}{2}+1 \leqslant \frac{r+s}{2}+1,$$

giving $r \geqslant s+2$. This is a contradiction. So H_C is non-special and by Riemann-Roch we get

$$s+2 \leqslant h^0(C, \mathcal{O}_C(H_C)) = d+1-g \leqslant r+s+1-g,$$

hence $g \leqslant r-1$.
(ii) Assume that $d \leqslant 2g$. We get by (i)

$$r+1 \leqslant s+2 \leqslant h^0(C, \mathcal{O}_C(H_C)) = d+1-g \leqslant g+1 \leqslant r,$$

which is absurd. □

Proposition 5. *Let $X \subset \mathbb{P}^n$ be smooth connected non-degenerate and linearly normal with $d \leqslant n$. Assume that the adjunction mapping $\varphi = \varphi_{|K+(r-1)H|}$ makes X a scroll over a smooth surface S. Then $S \simeq \mathbb{P}^2$ and X is one of the following:*
$r=4$*,* $d=10$*,* $X \simeq \mathbb{P}(T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$*, or*
$r=4$*,* $d=11$*,* $X \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$*, or* $r=5$*,* $d=10$*,* $X \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 4}(1))$*, i.e.* X *is the Segre embedding of* $\mathbb{P}^2 \times \mathbb{P}^3$*.*

Proof. Let S' be the smooth surface $X \cap H_1 \cap \cdots \cap H_{r-2}$, where H_i are generic hyperplanes in $\mathbb{P}^n$. We first remark that the geometric genus of S' is zero. This follows from Lemma 4 (ii) and the adjunction formula for $H_{S'}$. The standard exact sequences

$$\begin{aligned}0 \longrightarrow \mathcal{O}_X(K+(r-2)H) &\longrightarrow \mathcal{O}_X(K+(r-1)H)\\ &\longrightarrow \mathcal{O}_H(K_H+(r-2)H_H) \longrightarrow 0\end{aligned}$$

together with Lemma 1.1 from [12] show that, in our case, $h^0(X, \mathcal{O}_X(K+(r-1)H)) = g-q$. So, we have $\varphi : X \to S \subset \mathbb{P}^{g-q-1}$. Let H_S be the generic hyperplane section of $S \subset \mathbb{P}^{g-q-1}$ and let $Y =: \varphi^{-1}(H_S)$. Note that Y is a scroll of dimension $r-1$ over the curve H_S; if we let d_Y be its degree, we get $d_Y = (K+(r-1)H)H^{r-1} = 2g-2$ by adjunction formula. Let m be the dimension of

the projective space spanned by Y inside $\mathbb{P}^n$ (denoted below by $\langle Y\rangle$). By Barth's theorem (see [2]) we must have $m \geqslant 2(r-1)-1$. We get, using Lemma 4 (i)

$$m \geqslant 2r-3 \geqslant 2(r-2) \geqslant 2(g-1) = d_Y.$$

So, by Corollary 3, it follows that $H_S \simeq \mathbb{P}^1$. The two-dimensional version of Theorem A shows that $q=0$ and one of the following holds:

1. $S=\mathbb{P}^2$, $g=\Delta=3$;
2. S is a scroll over $\mathbb{P}^1$;
3. S is the Veronese embedding $v_2(\mathbb{P}^2)$, $g=6$.

Recalling the definition of $\Delta = d+r-h^0(X,\mathcal{O}_X(H))$, we get

$$n+r \geqslant d+r \geqslant n+1+\Delta,$$

giving $r \geqslant \Delta+1$. Now, if we are in case 1, by Proposition 4.7 from [12], it follows that we have the following possibilities for X:

$r=4$, $d=9,10$ or 11;

$r=5$, $d=10$, X is the Segre embedding of $\mathbb{P}^2\times\mathbb{P}^3$.

Assume that $r=4$, so $X\simeq\mathbb{P}(\mathcal{E})$ for some very ample vector bundle of rank three over $\mathbb{P}^2$. If ℓ is a line in $\mathbb{P}^2$, it follows that $\mathcal{E}|_\ell$ has degree 4 and is very ample. So, $\mathcal{E}|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(2)$, i.e. $\mathcal{E}$ is uniform. One may use the classification from [5]; we find that case $d=9$ is not possible, while for $d=10$ we get $\mathcal{E}\simeq T_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(1)$ (equivalently X is the hyperplane section of the Segre embedding of $\mathbb{P}^2\times\mathbb{P}^3$) and for $d=11$ we get $\mathcal{E}\simeq\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}(2)$ (this is the blowing-up of $\mathbb{P}^4$ with center a line).

To finish the proof we only have to show that cases 2 and 3 cannot occur. We use the notation from [11], Chapter V, Section 2. If we are in case 2, we have $S\simeq\mathbb{F}_e$, $H_S = C_0+bF$ with $b>e\geqslant 0$.

We look at the $(r-1)$-dimensional rational scrolls $Y_0=\varphi^{-1}(C_0)$ and $Y_1=\varphi^{-1}(F)$. If we put $d_i=\deg(Y_i)$ for $i=0,1$, we get $d_i\geqslant r-1$. Indeed, by Barth's theorem ([2]), if $m_i=\dim\langle Y_i\rangle$, we get $m_i\geqslant 2(r-1)-1$; moreover, since $\Delta(Y_i,\mathcal{O}_{Y_i}(H))=0$, we deduce

$$d_i+r-1 = h^0(Y_i,\mathcal{O}_{Y_i}(H)) \geqslant m_i+1 \geqslant 2(r-1),$$

i.e. $d_i\geqslant r-1$. So, we find

$$2g-2 = \deg(Y) = d_0+bd_1 \geqslant d_0+d_1 \geqslant 2(r-1),$$

contradicting part (i) of Lemma 4. Case 3 is ruled out by a similar argument. □

Next we need a general lemma concerning the geometry of hyperquadric fibrations (see also [13], 6.2).

Lemma 6. *Assume that the adjunction mapping $\varphi : X \to C \subset \mathbb{P}^m$ makes X a hyperquadric fibration over the smooth curve C. Then $m = g - q - 1$ and q coincides with the genus of C. Moreover, if we let $\mathcal{E} =: \varphi_* \mathcal{O}_X(H)$, $\mathcal{E}$ is a spanned vector bundle of rank $r+1$ over C. Denote by $\pi : \mathbb{P}(\mathcal{E}) \to C$ the projection and by L the tautological divisor on $\mathbb{P}(\mathcal{E})$. Then X is embedded in $\mathbb{P}(\mathcal{E})$ such that $L|_X = H$ and $X \in |2L + \pi^* B|$ for some divisor B on C. Finally, if $a =: \deg(\mathcal{E})$ and $b =: \deg(B)$, the following formulae hold*

$$a = 1 - g + 2(q-1) + d \quad \textit{and} \quad b = 2(g-1) - 4(q-1) - d.$$

Proof. From Lemma 1.1 in [12] and the standard exact sequences

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_X(K + (r-2)H) \longrightarrow \mathcal{O}_X(K + (r-1)H) \\ \longrightarrow \mathcal{O}_H(K_H + (r-2)H_H) \longrightarrow 0 \end{aligned}$$

it follows in our case that $h^0(X, \mathcal{O}_X(K + (r-1)H)) = g - q$. We have, for any $c \in C$, $H^0(X_c, \mathcal{O}_{X_c}(H)) = r + 1$ and $H^1(X_c, \mathcal{O}_{X_c}(H)) = 0$, so the existence of $\mathcal{E}$ follows from the base-change theorem. Moreover, the canonical diagram

$$\begin{array}{ccc} H^0(C, \mathcal{E}) & \xrightarrow{\sim} & H^0(X, \mathcal{O}_X(H)) \\ {\scriptstyle \mathrm{ev}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}} \\ \mathcal{E}_c & \xrightarrow{\sim} & H^0(X_c, \mathcal{O}_{X_c}(H)) \end{array}$$

shows that $\mathcal{E}$ is spanned by global sections iff the restriction map res is surjective for any $c \in C$. This holds true since X_c is a hyperquadric, hence linearly normal in $\mathbb{P}^r = \langle X_c \rangle$.

Consider also the canonical induced diagram

$$\begin{array}{ccc} X & \subset & \mathbb{P}(\mathcal{E}) \\ & {\scriptstyle \varphi}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & C & \end{array}$$

and write $X \sim 2L + \pi^* B$, for some $B \in \mathrm{Div}(C)$. Let H_C be the hyperplane section of $C \subset \mathbb{P}^{g-q-1}$. We find

$$\varphi^*(H_C) = K + (r-1)H = (K_{\mathbb{P}(\mathcal{E})} + X + (r-1)L)|_X = \varphi^*(K_C + \det \mathcal{E} + B).$$

By taking degrees, we get $g - 1 = 2(q-1) + a + b$. Moreover, $a = (L^{r+1})$, so $d = (L^r \cdot X) = 2a + b$. The two formulae follow. $\square$

Lemma 7. *Let $X \subset \mathbb{P}^n$ be smooth connected non-degenerate with $d \leqslant n$. Assume that the adjunction mapping $\varphi : X \to C$ makes X a hyperquadric fibration over the smooth curve C. Then $C \simeq \mathbb{P}^1$.*

Proof. Assume that $q = g(C) > 0$. By Lemma 4 (ii), $d \geqslant 2g + 1$. So, by Lemma 6, we have $b = 2(g-1) - d - 4(q-1) < 0$.

We show first that $\mathcal{E}$ is ample. As $\mathcal{E}$ is spanned, $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(L)$ is spanned. So, if L is not ample, there is a curve $D \subset \mathbb{P}(\mathcal{E})$ such that $(L \cdot D) = 0$. It follows that $(X \cdot D) = (2L + \pi^* B)D = \alpha b$ for some $\alpha > 0$. As $b < 0$, we deduce that $(X \cdot D) < 0$, so $D \subset X$. But $L|_X = H$, so $(D \cdot L) > 0$ which is a contradiction. So $\mathcal{E}$ is ample.

Let now $S \subset X$ be the surface-section of X, i.e. $S = X \cap H_1 \cap \cdots \cap H_{r-2}$, where H_i are generic hyperplanes in $\mathbb{P}^n$. We have $(H_S + K_S)^2 = 0$, giving $d + 2(H_S \cdot K_S) + (K_S)^2 = 0$. Adjunction formula yields $(H_S \cdot K_S) = 2g - 2 - d$; moreover, $(K_S)^2 \leqslant 8(1-q)$, since S is birationally ruled. We deduce, using also Lemma 4 (ii)

$$4(g-1) \geqslant d + 8(q-1) \geqslant 2g + 1 + 8(q-1).$$

So we get $4q \leqslant g+1$. By Lemma 6, $a = 1-g+2(q-1)+d$ and we find $a \leqslant d-2q$. Now, since $\mathcal{E}$ is ample and spanned, we may apply Proposition 2 to find

$$a = \deg(\mathcal{E}) \geqslant h^0(C, \mathcal{E}) = h^0(X, \mathcal{O}_X(H)) \geqslant n + 1.$$

Putting things together, we get

$$n + 1 \leqslant a \leqslant d - 2q \leqslant n - 2q.$$

This is a contradiction, so $q = 0$. □

We shall also need the proposition below which might have an interest in itself.

Proposition 8. *Let $X \subset \mathbb{P}^n$ be smooth, connected, non-degenerate and linearly normal. Assume that the adjunction mapping $\varphi : X \to C$ makes X a hyperquadric fibration over $C \simeq \mathbb{P}^1$. Assume moreover, that $d \geqslant 2g + 2$ and $r \geqslant g + 1$. Then, in the notation of Lemma* 6 *and denoting by $e = (e_0, \ldots, e_r)$ the splitting type of $\mathcal{E}$ and by F a fibre of the projection $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$, we have one of the following:*

(a) *$r = s$, $d = 2r$, $e = (1, \ldots, 1, 0)$, $X \in |2L|$;*

(b) *$r = s - 1$, $d = 2r + 1$, $e = (1, \ldots, 1)$, $X \in |2L - F|$;*

(c) *$r = s - 1$, $d = 2r$, $e = (1, \ldots, 1)$, $X \in |2L - 2F|$ or, equivalently, $X \simeq \mathbb{P}^1 \times Q^{r-1}$ embedded Segre;*

(d) *$r = s - 2$, $d = 2r + 2$, $e = (1, \ldots, 1, 2)$, $X \in |2L - 2F|$;*

(e) *$r = 3$, $X \simeq \mathbb{P}^1 \times \mathbb{F}_1$, embedded Segre, where $\mathbb{F}_1$ is embedded in $\mathbb{P}^4$ as a rational scroll of degree* 3*.*

Moreover, all these cases do exist.

Proof. We first remark that $g \geqslant 2$ (see [12]), so $r \geqslant 3$. Let Q denote a fibre of φ. We have $(H-Q)H^{r-1} = d-2$. The standard exact sequence

$$0 \longrightarrow \mathcal{O}_X(-Q) \longrightarrow \mathcal{O}_X(H-Q) \longrightarrow \mathcal{O}_H(H-Q) \longrightarrow 0$$

and the fact that $H^1(X, \mathcal{O}_X(-Q)) = 0$ allow one to prove by induction on r that $|H-Q|$ is base-points free. Note that on the curve-section of X, the degree of the restriction of $|H-Q|$ is $\geqslant 2g$, so it is base-points free. Moreover, $|H-Q|$ is not composed with a pencil, since $r \geqslant 3$. So, by Bertini's theorem, there is a smooth member $X' \in |H-Q|$. We let

$$\begin{aligned} H' &= H|_{X'}, & K' &= K_{X'}, \\ r' &= \dim(X') = r-1, & \varphi' &= \varphi_{|K'+(r'-1)H'|}, \\ d' &= \deg(X') = d-2, & g' &= g(H'), \\ s' &= h^0(X', \mathcal{O}_{X'}(H)) - 1 - r'. \end{aligned}$$

One finds easily $g' = g-1$, $s' = s-1$ and φ' identifies to $\varphi|_{X'}$. The statement of the proposition is proved by induction on r (note that we still have $d' \geqslant 2g'+2$ and $r' \geqslant g'+1$). Assume first that $g \geqslant 3$. Since $r \geqslant g+1$, for $r=4$ we get $g=3$ and we may use the classification from Theorem 4.3 in [12]. For $r \geqslant 4$ we find inductively the following possible values for the numerical invariants:

(a) $r = s$, $d = 2r$, $g = r-1$;
(b) $r = s-1$, $d = 2r+1$, $g = r-1$;
(c) $r = s-1$, $d = 2r$, $g = r-2$;
(d) $r = s-2$, $d = 2r+2$, $g = r-1$.

It remains to analyse the case $g = 2$, where one may use the classification theorem 3.4 in [12]. This leads to only one new case, which is (e).

Next we investigate the structure of $\mathcal{E}$ in each case.

First we have that $\mathcal{E}$ is non-special (since it is spanned by Lemma 6). So Riemann-Roch theorem gives

$$r+s+1 = h^0(\mathcal{E}) = a+r+1,$$

hence $a = s$. Now, in case (a), we remark that $|H-2Q| = \emptyset$, since $(H-Q)^{r-1} \cdot (H-2Q) = d-2r-2 < 0$. By Lemma 6, $b = 0$, so $X \in |2L|$.

The exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-L-2F) \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(L-2F) \longrightarrow \mathcal{O}_X(H-2Q) \longrightarrow 0$$

shows that $h^0(\mathcal{E}(-2)) = 0$; as $\mathcal{E}$ is spanned and $a = r$, the splitting-type of $\mathcal{E}$ must be $(1, \ldots, 1, 0)$. The existence follows by the same type of argument as in

the proof of Proposition 3 from [15]. The other cases are similar and simpler. For instance, in case (b) one gets as above $h^0(\mathcal{E}(-2)) = 0$, $a = r+1$ and $b = -1$. So $e = (1, \dots, 1)$, $\mathcal{E}$ is very ample and the existence follows now easily. □

Proposition 9. *Let $X \subset \mathbb{P}^n$ be smooth, connected, non-degenerate and linearly normal, with $d \leqslant n$. Assume that the adjunction mapping makes X a hyperquadric fibration over a smooth curve C. Then X is as in case* (ii) (b) *or case* (iv) *of the main theorem.*

Proof. By Lemma 7 $C \simeq \mathbb{P}^1$. We have $d \geqslant 2g+1$ and $g \leqslant r-1$ by Lemma 4. If $d \geqslant 2g+2$, we may apply Proposition 8, thus leading to cases (ii) (b) and (iv) (b) up to (iv) (e) of the main theorem. So, assume that $d = 2g+1$. As in the proof of Proposition 8 we deduce that $a = s$. By Lemma 6 we get $a = g$, $b = 1$. It follows $s = g \leqslant r-1$. Barth's theorem ([2]) ensures that $s \geqslant r-1$, so we must have $s = r-1$. We obtain

$$g = r-1, \quad d = 2r-1, \quad a = r-1.$$

As in the proof of Proposition 8, we have $|H - 2Q| = \emptyset$, so $h^0(\mathcal{E}(-2)) = 0$. It follows that the splitting type of $\mathcal{E}$ is $(1, \dots, 1, 0, 0)$, so we are in case (iv) (a) of the main theorem. The existence follows from Proposition 3 in [15]. □

We are now ready for the proof of our theorem.

Assume first that $r \leqslant s+1$. We have

$$\Delta = d + r - h^0(X, \mathcal{O}_X(H)) \leqslant n + r - n - 1 = r - 1.$$

If $\Delta = 0$, by Theorem A we get either case (iii) of the main theorem or some special examples of case (i). Similarly if $\Delta = 1$, by Theorem B we get either case (ii) (a) or some special examples of case (i). So, assume $\Delta \geqslant 2$, hence $r \geqslant 3$, from now on. If $r = 3$, it follows $\Delta = 2$, $s \geqslant 2$ and $\varphi : X \to \mathbb{P}^1$ is a hyperquadric fibration by [12], Theorem 3.12 and Corollary 3.3. If $r = 4$, we get $\Delta = 2$ or 3, $s \geqslant 3$, so φ is either a hyperquadric fibration over a rational curve or a scroll over $\mathbb{P}^2$ (see [12], Theorems 3.12, 4.8 and 4.2). Since $d \leqslant n$, it follows that $d \leqslant r + s \leqslant 2s+1$. So, using the general properties of the adjunction mapping (see e.g. [4], Chapters 9-11, in particular Theorem 11.2.4) and the above analysis for $r \leqslant 4$, it follows from Theorem I in [14] that one of the following holds:

(1) X is a scroll over a (smooth) curve C;
(2) φ makes X a scroll over a smooth surface;
(3) φ makes X a hyperquadric fibration over a smooth curve.

In case (1), from Corollary 3, we get $C \simeq \mathbb{P}^1$, so $\Delta = 0$. In case (2), by Proposition 5 we reach case (ii) (c). If we are in case (3), by Proposition 9 we get

case (ii) (b) or case (iv). Assume now that $r \geqslant s+2$. By Barth's theorem ([2]) it follows that $\mathrm{Pic}(X) \simeq \mathbb{Z}$, generated by the class of $\mathcal{O}_X(H)$. We show that X is Fano, so we are in case (i) and the main theorem is completely proved. As we have $\mathrm{Pic}(X) \simeq \mathbb{Z}$, to prove that X is Fano it is enough to see that the geometric genus of X, denoted p_g, is zero. Here we make use of a theorem of Harris (see [10]), generalising Castelnuovo's bound for the genus of a curve to arbitrary dimension.

It states that

$$\mathrm{p_g} \leqslant \binom{M}{r+1} s + \binom{M}{r} \varepsilon,$$

where $M = [(d-1)/s]$ and $\varepsilon = d - 1 - Ms$.

If $s = 1$ we find $\mathrm{p_g} = 0$ by an obvious direct computation. If $s \geqslant 2$ and $r \geqslant 2$ we get $r + s - 1 < rs$; our hypothesis $d \leqslant r + s$ gives $d - 1 < rs$, or $M < r$. So $\mathrm{p_g} = 0$. $\square$

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